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On a class of topological quantum field theories in three-dimensions

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ABSTRACT

We investigate the Chung-Fukuma-Shapere theory, or Kuperberg theory, of three-dimensional lattice topological field theory. We construct a functor which satisfies the Atiyah's axioms of topological quantum field theory by reformulating the theory as Turaev-Viro type state-sum theory on a triangulated manifold. The theory can also be extended to give a topological invariant of manifolds with boundary.

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1 Introduction

Many examples of topological field theory or topological invariants have been constructed. Some of them satisfy the axioms of topological quantum field theory given by Atiyah [1]. In two dimensions, we know the classification of the manifolds well, which leads to the complete classification of unitary topological quantum field theories on compact oriented manifolds [2]. On the other hand, the classification of topological field theories of the dimension $d \geq 3$ has not been done yet because of the difficulty of the classification of d-dimensional manifolds.

In three dimensions, it is known that pure gravity theory can be interpreted as the Chern-Simons-Witten theory [3] or the Turaev-Viro theory [4], which are both topological theories. Therefore the investigation of the three-dimensional topological field theories for a classification of them is important both from mathematical and physical point of view.

There are several ways of constructing topological field theories in three-dimensions such as 'surgery' method [5,6], 'state-sum' method [4,7–9] and, though it may not be well defined, the method using functional integral on a manifold M [10].

Among many such methods, we concentrate on a state-sum model, in particular, the Chung-Fukuma-Shapere theory [7]. The Chung-Fukuma-Shapere theory gives an invariant of closed 3-manifold M for each involutory Hopf algebra A. It is calculated explicitly by choosing a lattice L of M which is a cell complex with some good property. Note that the theory is equivalent to the invariant given by Kuperberg [8] which is defined on the basis of triangulations or Heegaard diagrams of oriented manifolds. The theory is well-defined only for a finite dimensional Hopf algebra A since it suffers divergences if we take an infinite dimensional one. Thus we set A to be finite dimensional.

It seems difficult to extend the Chung-Fukuma-Shapere invariant to a topological quantum field theory satisfying the axioms of Atiyah in its original form.[2] But the situation changes if we re-express the invariant as a form similar to the Turaev-Viro invariant by limiting a lattice to a simplicial complex, or triangulation. The Turaev-Viro theory gives the invariant of closed 3-manifolds and the functor which satisfies Atiyah's axioms.

In this paper, we explicitly construct the functor of topological quantum field theory by defining a weight, a correspondence of 'q-$6j$-symbol,' for a triangulated manifold. The

[2] Kuperberg announced in ref. [11] that his invariant can be extended to give the Atiyah's topological quantum field theory.

method of construction of the functor we use is similar to that of the Turaev-Viro theory.

We also show that the invariant can be extended to that of manifolds with boundary. It means that we give a complex number $\tilde{F}_A(M)$ which is determined only by the topology of M to each compact manifold M. Our theory gives an extended version of Chung-Fukuma-Shapere invariant whereas the theory by Karowski, Müller and Schrader in ref. [12] gives that of the Turaev-Viro invariant. This fact also shows a similarity between the Chung-Fukuma-Shapere invariant and the Turaev-Viro invariant.

2 The Chung-Fukuma-Shapere Theory: Invariant of Closed 3-manifolds

Let M be a closed 3-manifold. The partition function of the Chung-Fukuma-Shapere theory, which is a topological invariant of M, $Z_A(M)$, is defined for each involutory Hopf algebra $(A; m, u, \Delta, \epsilon, S)$ over $\mathbf{C}$ [7]. An involutory Hopf algebra is a Hopf algebra with the property that square of the antipode operator is identity : $S^2 = id.$.

We explicitly write the operations on A by means of a basis $\{\phi_x | x \in X\}$ of A as follows :

$$\begin{aligned} m(\phi_x \otimes \phi_y) &= \sum_{z\in X} C_{xy}{}^z \phi_z, & (1)\\ u(1) &= \sum_{x\in X} u^x \phi_x, & (2)\\ \Delta(\phi_x) &= \sum_{y,z\in X} \Delta_x{}^{yz} \phi_y \otimes \phi_z, & (3)\\ \epsilon(\phi_x) &= \epsilon_x, & (4)\\ S(\phi_x) &= \sum_{y\in X} S^y{}_x \phi_y, & (5)\end{aligned}$$

where $C_{xy}{}^z, u^x, \epsilon_x, 1 \in \mathbf{C}$. Symbols m, u, Δ, ϵ and S denote multiplication, unit, comultiplication, counit and antipode respectively. From now on, we often assume that the repeated indices are summed over.

We define the metric g_{xy} and the cometric h^{xy} by

$$g_{xy} \equiv C_{xu}{}^v C_{yv}{}^u, \qquad h^{xy} \equiv \Delta_u{}^{vx} \Delta_v{}^{uy}. \tag{6}$$

Since A is involutory, g_{xy} and h^{xy} have inverses g^{xy} and h_{xy}. We use g_{xy} and g^{xy} to raise

and lower the indices of $C_{xy}{}^z$ and u^x (e.g., $C_{xyz} = g_{zu}C_{xy}{}^u$). Similarly, we use h^{xy} and h_{xy} for $\Delta_x{}^{yz}$ and ϵ_x.

We summarize some important relations among C, Δ and S which hold generally for an involutory Hopf algebra. Some of these relations play an important role in verifying the topological invariance of $Z_A(M)$.

$$C_{x_1x_2\cdots x_n} = C_{x_nx_1\cdots x_{n-1}} \,, \tag{7}$$

$$\Delta^{x_1x_2\cdots x_n} = \Delta^{x_nx_1\cdots x_{n-1}} \,, \tag{8}$$

$$S^x{}_y = |X|^{-1}g^{xz}h_{zy} \,, \tag{9}$$

$$|X|^{-1}C_{x_1x_2\cdots x_n}\,C_{y_1y_2\cdots y_n}\,\Delta_{z_1}{}^{x_1y_1}\,\Delta_{z_2}{}^{x_2y_2}\,\cdots\,\Delta_{z_n}{}^{x_ny_n} = C_{z_1z_2\cdots z_n} \,, \tag{10}$$

$$C_{x_1x_2\cdots x_n}\,S^{x_1}{}_{y_1}\,S^{x_2}{}_{y_2}\,\cdots\,S^{x_n}{}_{y_n} = C_{y_n\cdots y_2y_1} \tag{11}$$

$$\Delta^{x_1x_2\cdots x_n}\,S^{y_1}{}_{x_1}\,S^{y_2}{}_{x_2}\,\cdots\,S^{y_n}{}_{x_n} = \Delta^{y_n\cdots y_2y_1} \tag{12}$$

where $|X|$ is an order of the algebra A and

$$C_{x_1x_2\cdots x_n} \;\equiv\; C_{a_1x_1}{}^{a_2}C_{a_2x_2}{}^{a_3}\times\cdots\times C_{a_{n-1}x_{n-1}}{}^{a_n}C_{a_nx_n}{}^{a_1} \,, \tag{13}$$

$$\Delta^{x_1x_2\cdots x_n} \;\equiv\; \Delta_{a_1}{}^{x_1a_2}\Delta_{a_2}{}^{x_2a_3}\times\cdots\times\Delta_{a_{n-1}}{}^{x_{n-1}a_n}\Delta_{a_n}{}^{x_na_1} . \tag{14}$$

With these preparations, we now recall the definition of the invariant $Z_A(M)$ given in ref. [7].

We first choose a lattice L which represents M. Here a lattice L is a three-dimensional cell complex such that every 2-cell is a polygon and every 1-cell is a boundary of at least three 2-cells. The definition is given as follows :

1. Decompose L into the set of polygonal faces $F = \{f_i\}_{i=1,\ldots,N_2}$ and that of hinges $H = \{h_i\}_{i=1,\ldots,N_1}$ as depicted in fig. 1. Here N_i denotes the number of i-cells in L. We pick an orientation of each face f_i and put an arrow on each edge according to it. We associate symbols $(i,1),(i,2),\ldots,(i,n)$ to the edges of each n-gon f_i (Fig.1).

2. Assign $C_{x_{(i,1)}x_{(i,2)}\cdots x_{(i,n)}} \in \mathbf{C}$ to each n-gonal face f_i where the index $x_{(i,j)}$ is an element of X.

3. The assignment of an arrow and a symbol to each edge of faces induces those to each edge of hinges $\{h_i\}$. The m arrows on the edges of a m-hinge h are not always in the same direction.

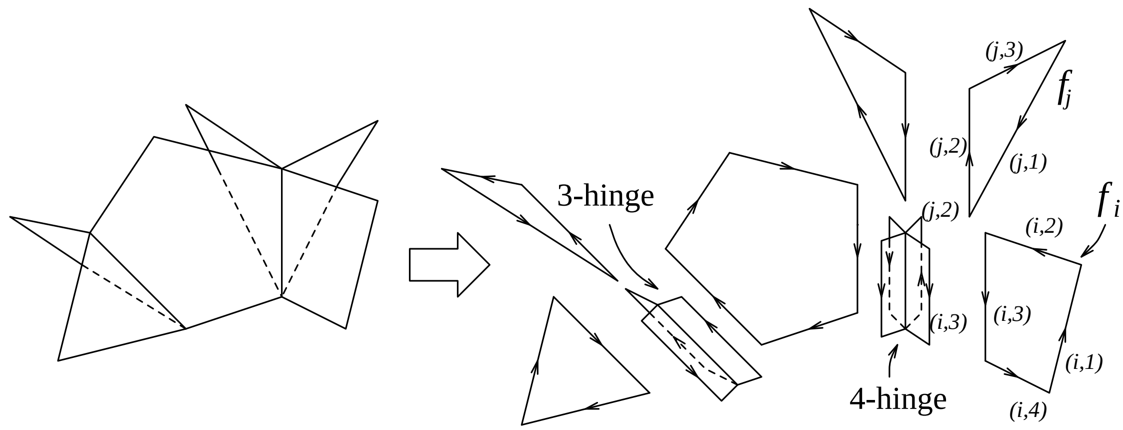

Figure 1: The decomposition of a part of a lattice L into faces and hinges. An n-hinge pastes n different faces. Arrows are laid on each edge of faces or hinges.

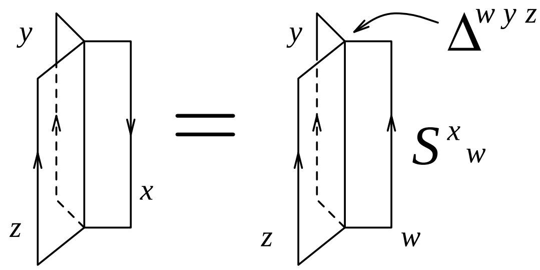

Figure 2: The operation of the direction changing operator S. The role of the hinge in the left-hand side is the same as that in the right-hand side : $\Delta^{wyz} \times S^x{}_w$.

If all arrows in the m-hinge h are in the same direction, we align the indices on the edges of the hinge associated as above in the clockwise order of the edges around the arrows as (i_1, j_1), (i_2, j_2), $\cdots$ (i_m, j_m) and associate $\Delta^{x_{(i_1,j_1)}x_{(i_2,j_2)}\cdots x_{(i_m,j_m)}} \in \mathbf{C}$ to h.

If directions of arrows on some edges of h are not the same as the rest, we change the direction of the arrows so as to make directions of all the arrows match by multiplying an additional factor (the direction changing operators) $S^x{}_{x'} \in \mathbf{C}$ (See fig. 2) for each edge of which we would like to upside down the direction of arrow.

4. So far we have defined the weight $C_{x_{(i,1)}\cdots x_{(i,n)}}$ for each face f_i and

$\Delta^{x_{(i_1,j_1)}\cdots x'_{j_k}\cdots x_{(i_m,j_m)}} \prod_{j_k \in R_h} S^{x_{(i_k,j_k)}}{}_{x'_{j_k}}$ for each hinge h. The set R_h corresponds to the set of all direction changing edges of a hinge h. The partition function is defined by contracting indices as

$$Z_A(L) = |X|^{-N_1-N_3} \prod_{i=1}^{N_2} C_{x_{(i,1)}\cdots x_{(i,n_i)}} \prod_{h\in H} \left[\Delta^{x_{(i_1,j_1)}\cdots x'_{j_k}\cdots x_{(i_m,j_m)}} \prod_{j_k \in R_h} S^{x_{(i_k,j_k)}}{}_{x'_{j_k}} \right]. \tag{15}$$

It is shown that this value does not depend on the direction of arrows on edges of each face nor any local deformation of a lattice L which preserves a topology of L. Thus,

Theorem 1 (Chung-Fukuma-Shapere [7]) *$Z_A(L)$ is a topological invariant of a manifold M.*

Note that in particular, if a lattice L is a simplicial complex, $Z_A(L)$ is invariant under Alexander moves of L [13].

3 Some Examples

We calculate Z_A for some manifolds M.

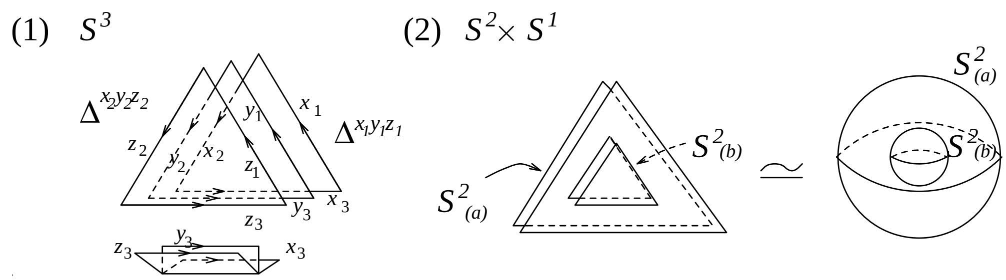

Figure 3: (1) S^3 : $N_3 = N_1 = 3$; (2) $S^2 \times S^1$: Outside of the sphere $S^2{}_{(a)}$ and inside of $S^2{}_{(b)}$ are identified. $N_3 = N_1 = 3$.

For a three-sphere S^3, we can take a lattice consisting of three triangles and three hinges (Fig.3(1)). The invariant Z_A is calculated as

$$\begin{aligned} Z_A(S^3) &= |X|^{-3-3} C_{x_1x_2x_3} C_{y_1y_2y_3} C_{z_1z_2z_3} \Delta^{x_1y_1z_1} \Delta^{x_2y_2z_2} \Delta^{x_3y_3z_3} \\ &= |X|^{-5} C_{w_1w_2w_3} C_{z_1z_2z_3} h^{z_1w_1} h^{z_2w_2} h^{z_3w_3} \end{aligned}$$

$$
\begin{aligned}
&= |X|^{-2}\, C_{z_1 z_2 z_3} C_{u_3 u_2 u_1} g^{z_1 u_1} g^{z_2 u_2} g^{z_3 u_3} \\
&= |X|^{-2}\, C_{z_1 z_2}{}^{u_3} C_{u_1 u_3}{}^{z_2} g^{z_1 u_1} \\
&= |X|^{-2}\, g_{z_1 u_1} g^{z_1 u_1} \\
&= |X|^{-1} \qquad (16)
\end{aligned}
$$

where we use the relations (6), (9), (10) and (11).

The next example is a manifold $S^2 \times S^1$. Considering the lattice in Fig.3(2), the invariant can be evaluated as

$$
\begin{aligned}
Z_A(S^2 \times S^1) &= |X|^{-3-3} C_{x_1 x_2 x_3} C_{y_1 y_2 y_3} C_{z_1 z_2 z_3} C_{w_1 w_2 w_3} \Delta^{x_1 y_1 z_1 w_1} \Delta^{x_2 y_2 z_2 w_2} \Delta^{x_3 y_3 z_3 w_3} \\
&= |X|^{-4} C_{u_1 u_2 u_3} C_{v_1 v_2 v_3} h^{u_1 v_1} h^{u_2 v_2} h^{u_3 v_3} \\
&= 1. \qquad (17)
\end{aligned}
$$

Here we use the relation $\Delta^{x_1 y_1 z_1 w_1} = \Delta_{u_1}{}^{x_1 y_1} \Delta_{v_1}{}^{z_1 w_1} h^{u_1 v_1}$.

Note that the values $Z_A(S^3)$ and $Z_A(S^2 \times S^1)$ are both independent of the choice of the algebra A.

4 Another Representation

In this section, we give another representation of the Chung-Fukuma-Shapere invariant. The idea is to take a simplicial complex, or a triangulation, as a lattice L and define a weight W_i, which acts similarly as a quantum $6j$-symbol in the case of the Turaev-Viro theory, on each tetrahedron T_i. This makes the invariant $Z_A(L)$ the form

$$
Z_A(L) = N \sum \prod_{i=1}^{N_3} W_i \qquad (18)
$$

where N is a normalization factor which will be given explicitly later.

Now we begin by making preparations for obtaining the form eq.(18) from eq.(15). Let M be a triangulated manifold, i.e., a simplicial complex representing a certain manifold. We number all the vertices of M arbitrarily as $1, 2, ..., N_0$, and according to that we associate an arrow along each 1-simplex (edge) as follows : for an edge whose boundary consists of vertices i and j, the direction of an arrow on the edge is $i \longrightarrow j$ if $i > j$, and

$i \longleftarrow j$ otherwise.[3] In this way, all tetrahedra T_i ($i \in \{1, 2, ..., N_3\}$) are divided into two classes, U^+ and U^-, according to the orientation of the order of four vertices a, b, c and d of T_i. We define the tetrahedra whose vertices are oriented as (a) (or (b)) of Fig.4 to be in a class U^+ (or class U^-) under the assumption that $a > b > c > d$.

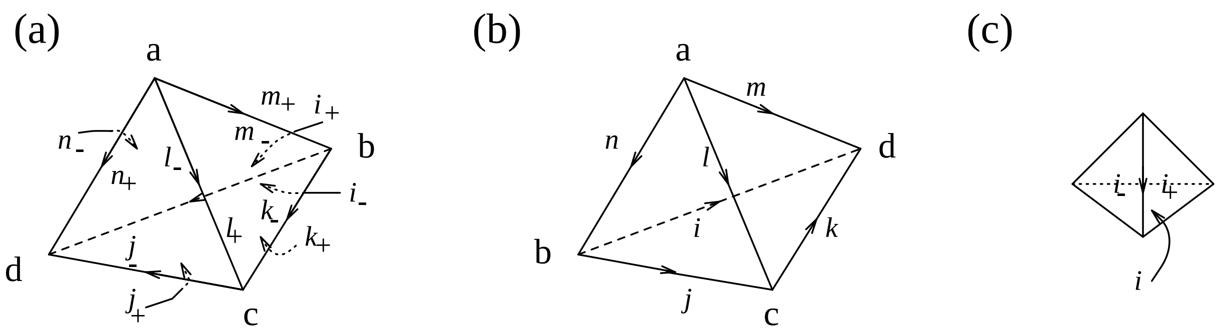

Figure 4: Tetrahedron of class (a) U^+ or (b) U^-, where $a > b > c > d$. We use the simplified notation in the figure (b). The rule is as same as (a), which is denoted in (c).

Next, we 'color' the triangulated manifold M by associating a symbol which is an element of X to each pair of a 1-simplex E_i and a 2-simplex F_j in M if $E_i \cap F_j = E_i$. This gives $3 \times N_2$ symbols on M. To be concrete, a coloring ϕ on M is a map

$$\phi : \{(E_i, F_j) \,|\, i = 1, \cdots, N_1,\ j = 1, \cdots, N_2,\ E_i \cap F_j = E_i\} \longrightarrow X. \tag{19}$$

Note that a coloring on M induces a coloring on each tetrahedron T_i (Fig.4).

Then, given an involutory Hopf algebra, we define a weight ${W_i}^{\kappa_i}$ on a tetrahedron T_i which is given coloring and arrows on edges as Fig.4 (a) or (b) :

$${W_i}^{\kappa_i} = \begin{cases} {W_i}^+ & \text{for} \quad T_i \in U^+ \\ {W_i}^- & \text{for} \quad T_i \in U^- \end{cases} \tag{20}$$

where

$$\begin{aligned} {W_i}^+ \begin{pmatrix} i_+, i_- & j_+, j_- & k_+, k_- \\ l_+, l_- & m_+, m_- & n_+, n_- \end{pmatrix} \quad &= \quad C_{I'KJ}\, C_{K'M'L}\, C_{J'L'N}\, C_{IN'M}\, {S^{I'}}_{I''}\, {\Delta^{I''Ii_+}}_{i_-} \\ &\times {S^{J'}}_{J''}\, {\Delta^{J''Jj_+}}_{j_-}\, {S^{K'}}_{K''}\, {\Delta^{K''Kk_+}}_{k_-}\, {S^{L'}}_{L''}\, {\Delta^{L''Ll_+}}_{l_-} \\ &\times {S^{M'}}_{M''}\, {\Delta^{M''Mm_+}}_{m_-}\, {S^{N'}}_{N''}\, {\Delta^{N''Nn_+}}_{n_-} \end{aligned} \tag{21}$$

[3] We choose such a way of defining the direction of arrows only for simplicity. In practice, we can define $Z_A(L)$ of (18) for a manifold whose arrows on 1-simplices are given arbitrarily, though it is rather complicated. We comment on the point later again.

and

$$W_i{}^{-}\begin{pmatrix} i_+,i_- & j_+,j_- & k_+,k_- \\ l_+,l_- & m_+,m_- & n_+,n_- \end{pmatrix} = C_{IK'J'}\,C_{KM'L}\,C_{JL'N}\,C_{I'N'M}\,S^{I'}{}_{I''}\,\Delta^{I''Ii_+}{}_{i_-}$$
$$\times S^{J'}{}_{J''}\,\Delta^{J''Jj_+}{}_{j_-}\,S^{K'}{}_{K''}\,\Delta^{K''Kk_+}{}_{k_-}\,S^{L'}{}_{L''}\,\Delta^{L''Ll_+}{}_{l_-}$$
$$\times S^{M'}{}_{M''}\,\Delta^{M''Mm_+}{}_{m_-}\,S^{N'}{}_{N''}\,\Delta^{N''Nn_+}{}_{n_-}. \tag{22}$$

The indices $I, I', I'', J, ...$ are elements of a set X and summed over in these equations. We define

$$F(M) \equiv |X|^{-2N_2-N_0}\sum_{\{\phi\}}\prod_{i=1}^{N_3} W_i{}^{\kappa_i} \in \mathbf{C} \tag{23}$$

where sum of ϕ is taken over all the maps satisfying (19).

Proposition 1 $F(M) = Z_A(M)$.

Proof.

First, we explicitly write down sum over colorings $\{\phi\}$ on M in eq.(23). We pay attention to a 1-simplex E_i and give numbers $1, 2, ..., m_i$ to all triangles which include E_i as an edge by a clockwise order with respect to the direction of the arrow on E_i. This induces a color x^i_k on a pair (E_i, F_k) where $k = 1, \cdots m_i$. Denoting all colorings of M as the same way, we rewrite the sum over all colorings $\{\phi\}$ as

$$\sum_{\{\phi\}} \to \prod_{i=1}^{N_1}\sum_{x^i_1,x^i_2,\cdots x^i_{m_i}\in X}. \tag{24}$$

Extracting a part which depends on the sum of $x^i_1, x^i_2, \cdots x^i_{m_i} \in X$ from eq.(23) by use of (21) and (22), and performing the calculation, we get for each 1-simplex i

$$\sum_{x^i_1,x^i_2,\cdots x^i_{m_i}\in X}\Delta^{z^i_1 y^i_1 x^i_2}{}_{x^i_1}\,\Delta^{z^i_2 y^i_2 x^i_3}{}_{x^i_2}\cdots\Delta^{z^i_{m_i} y^i_{m_i} x^i_1}{}_{x^i_{m_i}} = \Delta^{z^i_1\,y^i_1\,z^i_2\,y^i_2\cdots z^i_{m_i}\,y^i_{m_i}}. \tag{25}$$

Thus eq.(23) is rewritten as

$$F(M) = |X|^{-N_1-N_2-N_3}\prod_{i=1}^{N_1}\sum_{z^i_l,y^i_l,y^{i+N_1}_l}\left[\Delta^{z^i_1 y^i_1 z^i_2 y^i_2\cdots z^i_{m_i} y^i_{m_i}}\,S^{y^{i+N_1}_1}{}_{z^i_1}\,S^{y^{i+N_1}_2}{}_{z^i_2}\cdots S^{y^{i+N_1}_{m_i}}{}_{z^i_{m_i}}\right]$$
$$\times\prod_{k=1}^{N_2} C_{y^{j_1}_{k_1}y^{j_2}_{k_2}\,y^{j_3}_{k_3}}\,C_{y^{j_{\alpha_1}}_{k_{\alpha_1}}\,y^{j_{\alpha_2}}_{k_{\alpha_2}}y^{j_{\alpha_3}}_{k_{\alpha_3}}}. \tag{26}$$

Here we use the Poincare duality theorem $N_3 - N_2 + N_1 - N_0 = 0$. The quantity $C_{y^{j_1}_{k_1}y^{j_2}_{k_2}\,y^{j_3}_{k_3}}\times\, C_{y^{j_{\alpha_1}}_{k_{\alpha_1}}\,y^{j_{\alpha_2}}_{k_{\alpha_2}}y^{j_{\alpha_3}}_{k_{\alpha_3}}}$ comes from two tetrahedra whose intersection is a triangle F_k

and $(k_{\alpha_1}, k_{\alpha_2}, k_{\alpha_3})$ is a permutation of (k_1, k_2, k_3). Thus we see that the value (26) is the Chung-Fukuma-Shapere invariant $Z_A(L)$ for a lattice L which is generated by gluing N_3 simplices $\{T_i\}_{i=1,\cdots N_3}$. Note that L is a lattice M whose faces are all duplicated. Thus it is different from M as a lattice and it has $N_3 + N_2$ 3-cells and $2 \times N_2$ 2-cells. Since $Z_A(L)$ is a topological invariant, it is the same as $Z_A(M)$. Thus the proof is completed. ■

Note that for a triangulated manifold M with arrows of arbitrary direction not induced from the order of the vertices, $F(M)$ can also be defined. In this case, since it is possible that there exists a 3-simplex T which belongs to neither U^+ nor U^-, we have to give a definition of W^κ for such T. For example, for a 3-simplex T whose arrows and weights are the same as $T_+ \in U^+$ except the arrow on an edge j is in the opposite direction, the weight is obtained by multiplying $S \cdot S$ as follows :

$$W^\kappa \begin{pmatrix} i_+, i_- & j_+, j_- & k_+, k_- \\ l_+, l_- & m_+, m_- & n_+, n_- \end{pmatrix} = \sum_{j'_+, j'_- \in X} W^+ \begin{pmatrix} i_+, i_- & j'_-, j'_+ & k_+, k_- \\ l_+, l_- & m_+, m_- & n_+, n_- \end{pmatrix} S_{j_- \, j'_-} \, S^{j_+ \, j'_+}. \tag{27}$$

The weight for any other 3-simplex can be obtained similarly. For simplicity, we write eq.(27) as

$$W^\kappa \begin{pmatrix} \bar{i}_\uparrow & \bar{j}_\uparrow & \bar{k}_\uparrow \\ \bar{l}_\uparrow & \bar{m}_\uparrow & \bar{n}_\uparrow \end{pmatrix} = W^+ \begin{pmatrix} \bar{i}_\uparrow & \bar{j}'_\downarrow & \bar{k}_\uparrow \\ \bar{l}_\uparrow & \bar{m}_\uparrow & \bar{n}_\uparrow \end{pmatrix} \bar{S}^{\bar{j}}{}_{\bar{j}'} \tag{28}$$

where the new symbols $\bar{i}_\uparrow$ and $\bar{i}_\downarrow$ stand for pairs (i_+, i_-) and (i_-, i_+) respectively and

$$\bar{S}^{\bar{j}}{}_{\bar{j}'} = S_{j_- \, j'_-} \, S^{j_+ \, j'_+}. \tag{29}$$

We can consider that $\bar{i}_\uparrow$ and $\bar{i}_\downarrow$ are the same colored 1-simplex but the directions of arrows are opposite to each other.

Now we give some properties of the weight W^κ for a tetrahedron T.

Remember that in the case of the Turaev-Viro theory the quantum $6j$-symbol $| ::: |$ [4] defined for a tetrahedron with admissible color on six edges of it has the symmetries

$$\begin{vmatrix} i & j & k \\ l & m & n \end{vmatrix} = \begin{vmatrix} i & m & n \\ l & j & k \end{vmatrix} = \begin{vmatrix} l & m & k \\ i & j & n \end{vmatrix} = \begin{vmatrix} l & j & n \\ i & m & k \end{vmatrix} = \begin{vmatrix} j & i & k \\ m & l & n \end{vmatrix} = \begin{vmatrix} i & k & j \\ l & n & m \end{vmatrix}. \tag{30}$$

[4] The theory can also be defined by using other 'symbol' than quantum $6j$-symbol which satisfies a certain property. [4]

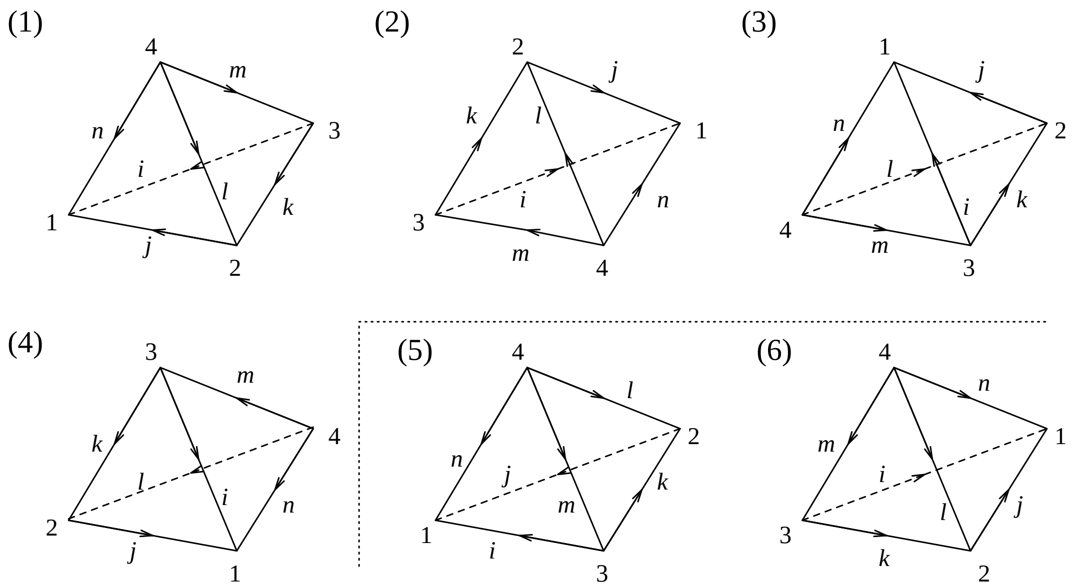

Figure 5: Symmetries of the weight W: (1)-(4) are the colored tetrahedra with the same orientation and others are those with the opposite orientation. Arrows are associated such that the tetrahedron (1) is the same as the (a) of Fig.4.

These symmetries correspond to the rotational symmetry and orientation changing symmetry of the tetrahedron, namely the symmetries among the six colored tetrahedra depicted in Fig.5. Here we use the term orientation as that of the tetrahedron as a 2-sphere S^2.

In our case of the weight W^κ, the relation (30) is not satisfied in its original form since each 1-simplex has an arrow in addition to the color. Instead, we have a modified version. In the case of a tetrahedron T of the class U^+ or U^-, it can be written as follows:

$$W^{\pm}\begin{pmatrix} \bar{i}_{\uparrow} & \bar{j}_{\uparrow} & \bar{k}_{\uparrow} \\ \bar{l}_{\uparrow} & \bar{m}_{\uparrow} & \bar{n}_{\uparrow} \end{pmatrix} \tag{31}$$

$$= \sum_{\bar{i}'\bar{k}'\bar{l}'\bar{n}'} W^{\pm}\begin{pmatrix} \bar{i}'_{\downarrow} & \bar{m}_{\uparrow} & \bar{n}'_{\downarrow} \\ \bar{l}'_{\downarrow} & \bar{j}_{\uparrow} & \bar{k}'_{\downarrow} \end{pmatrix} \bar{S}^{\bar{i}}{}_{\bar{i}'}\bar{S}^{\bar{k}}{}_{\bar{k}'}\bar{S}^{\bar{l}}{}_{\bar{l}'}\bar{S}^{\bar{n}}{}_{\bar{n}'} \tag{32}$$

$$= \sum_{\bar{i}'\bar{j}'\bar{k}'\bar{l}'\bar{m}'\bar{n}'} W^{\pm}\begin{pmatrix} \bar{l}'_{\downarrow} & \bar{m}'_{\downarrow} & \bar{k}'_{\downarrow} \\ \bar{i}'_{\downarrow} & \bar{j}'_{\downarrow} & \bar{n}'_{\downarrow} \end{pmatrix} \bar{S}^{\bar{i}}{}_{\bar{i}'}\bar{S}^{\bar{j}}{}_{\bar{j}'}\bar{S}^{\bar{k}}{}_{\bar{k}'}\bar{S}^{\bar{l}}{}_{\bar{l}'}\bar{S}^{\bar{m}}{}_{\bar{m}'}\bar{S}^{\bar{n}}{}_{\bar{n}'} \tag{33}$$

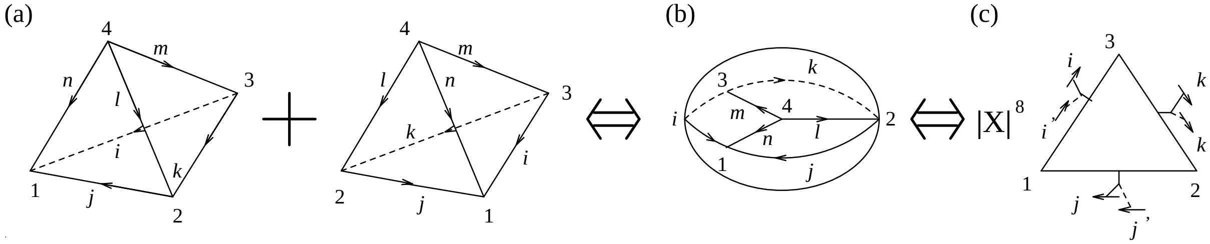

Figure 6: The relation (35): Connection of two tetrahedra in (a) by three edges l, m and n induces (b), which gives the same weight as that of (c).

$$= \sum_{\bar{j}'\bar{m}'} W^{\pm} \begin{pmatrix} \bar{l}_{\uparrow} & \bar{j}'_{\downarrow} & \bar{n}_{\uparrow} \\ \bar{i}_{\uparrow} & \bar{m}'_{\downarrow} & \bar{k}_{\uparrow} \end{pmatrix} \bar{S}^{\bar{j}}{}_{\bar{j}'} \bar{S}^{\bar{m}}{}_{\bar{m}'}. \tag{34}$$

where eqs.(31)-(34) correspond to the tetrahedra (1)-(4) in Fig.5 respectively. Note that the above relations are only among tetrahedra with the same orientation. Generically, there is no corresponding symmetry between two tetrahedra of different orientation.

The weight also obeys the relation

$$\begin{aligned} & W^{+} \begin{pmatrix} i_+, i & j, j_- & k, k_- \\ l_+, l_- & m_+, m_- & n_+, n_- \end{pmatrix} W^{+} \begin{pmatrix} k_-, k' & j_{0+}, j_{0-} & i', i_+ \\ n_-, n_+ & m_-, m_+ & l_-, l_+ \end{pmatrix} S_{j_{0+}j'} S^{j_{0-}j_-} \\ = \; & W^{+} \begin{pmatrix} i_+, i & j, j_- & k, k_- \\ l_+, l_- & m_+, m_- & n_+, n_- \end{pmatrix} W^{-} \begin{pmatrix} i', i_+ & k_-, k' & j_-, j' \\ l_-, l_+ & n_-, n_+ & m_-, m_+ \end{pmatrix} \\ = \; & |X|^8 C_{IKJ} \Delta^{Ii_0}{}_{i'_0} S_{i_0 i} S^{i'_0 i'} \Delta^{Kk}{}_{k'} \Delta^{Jj}{}_{j'}. \end{aligned} \tag{35}$$

From this equation we obtain the orthogonality relation analogous to that of quantum $6j$-symbol case :

$$\begin{aligned} & W^{+} \begin{pmatrix} i_+, i_- & j, j_- & k_+, k_- \\ l_+, l_- & m_+, m_- & n_+, n_- \end{pmatrix} W^{+} \begin{pmatrix} k_-, k_+ & j_{0+}, j_{0-} & i_-, i_+ \\ n_-, n_+ & m_-, m_+ & l_-, l_+ \end{pmatrix} S_{j_{0+}j'} S^{j_{0-}j_-} \\ = \; & W^{+} \begin{pmatrix} i_+, i_+ & j, j_- & k_+, k_- \\ l_+, l_- & m_+, m_- & n_+, n_- \end{pmatrix} W^{-} \begin{pmatrix} i_-, i_+ & k_-, k_+ & j_-, j' \\ l_-, l_+ & n_-, n_+ & m_-, m_+ \end{pmatrix} \\ = \; & |X|^{10} \delta^{j}{}_{j'}. \end{aligned} \tag{36}$$

We also have analogous relations to the Biedenharn-Elliot identities of the quantum $6j$-

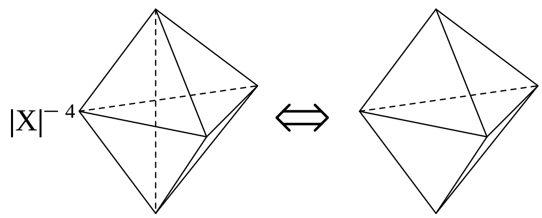

Figure 7: The relation (37). It corresponds to the topology preserving move, $(3,2)$-move.

symbols :

$$
\begin{aligned}
&|X|^{-4}W^{+}\begin{pmatrix} n_1,n_2 & j_3,j_2 & l_3,l_2 \\ q_1,q_2 & p_1,p_3 & r_1,r_3 \end{pmatrix} \\
&\qquad\times W^{+}\begin{pmatrix} i_3,i_2 & j_1,j_3 & k_1,k_2 \\ l_1,l_3 & m_3,m_2 & n_2,n_3 \end{pmatrix} W^{-}\begin{pmatrix} n_3,n_1 & m_1,m_3 & i_1,i_3 \\ s_1,s_2 & r_3,r_2 & p_3,p_2 \end{pmatrix} \\
&= W^{+}\begin{pmatrix} i_1,i_2 & j_1,j_2 & k_1,k_3 \\ q_3,q_2 & s_1,s_3 & r_1,r_2 \end{pmatrix} W^{-}\begin{pmatrix} l_1,l_2 & m_1,m_2 & k_3,k_2 \\ s_3,s_2 & q_1,q_3 & p_1,p_2 \end{pmatrix} \quad . \qquad (37)
\end{aligned}
$$

By changing the direction of arrows on some 1-simplices by multiplying $S\cdot S$, we obtain an equation of another type, for example,

$$
\begin{aligned}
&|X|^{-4}W^{+}\begin{pmatrix} t_3,t_2 & u_1,u_3 & j_1,j_2 \\ n_1,n_3 & l_3,l_2 & r_1,r_2 \end{pmatrix} \\
&\qquad\times W^{+}\begin{pmatrix} s_3,s_2 & t_1,t_3 & k_1,k_2 \\ l_1,l_3 & m_3,m_2 & r_2,r_3 \end{pmatrix} W^{-}\begin{pmatrix} s_1,s_3 & i_1,i_2 & u_3,u_2 \\ n_3,n_2 & r_3,r_1 & m_1,m_3 \end{pmatrix} \\
&= W^{+}\begin{pmatrix} i_3,i_2 & j_3,j_2 & k_3,k_2 \\ l_1,l_2 & m_1,m_2 & n_1,n_2 \end{pmatrix} W^{+}\begin{pmatrix} t_1,t_2 & u_1,u_2 & j_1,j_3 \\ i_1,i_3 & k_1,k_3 & s_1,s_2 \end{pmatrix} \quad . \qquad (38)
\end{aligned}
$$

5 Manifolds with Boundary

From now on, we consider an compact triangulated 3-manifold M with an arrow along each 1-simplex. We denote M_i $(i=0,1,2)$ the number of i-simplices of the boundary of M : ∂M.

Let ϕ be a coloring of M as (19). In particular, we call $\partial\phi$ a coloring of ∂M :

$$\partial\phi : \{(E_i,F_j)\,|\,E_i,F_j\in\partial M\,,\,E_i\cap F_j=E_i\}\longrightarrow X. \qquad (39)$$

For a fixed coloring $\partial\phi$ on the boundary ∂M and an involutory Hopf algebra A, we define

$$F(M;\partial\phi) = |X|^{-2N_2-N_0+(2M_2+M_0)/2} \sum_{\{\phi\}|_{\partial\phi}} \prod_{i=1}^{N_3} W_i^{\kappa_i} \in \mathbf{C}. \tag{40}$$

Here we denote $\{\phi\}|_{\partial\phi}$ the set of all colorings $\{\phi\}$ with a fixed coloring $\partial\phi$ on ∂M. Note that in the case of a closed manifold, eq.(40) reduces to the Chung-Fukuma-Shapere invariant eq.(15) or eq.(23).

Proposition 2 *$F(M;\partial\phi)$ does not depend on the direction of an arrow on a 1-simplex in $M \setminus \partial M$ and it is invariant under local topology preserving deformation of $M \setminus \partial M$.*

The proof of this proposition is straightforward from proposition 1 and the note under the theorem 1 since we know that all topology preserving deformation of $M \setminus \partial M$ is generated by a finite sequence of Alexander moves and their inverses.

6 Construction of the Functor

In this section, we interpret $F(M;\partial\phi)$ as the operator on a boundary of M and construct a functor which satisfies Atiyah's axioms of 3-dimensional topological quantum field theory [1]. The construction can be done along lines similar to ref. [4].

A 3-dimensional cobordism (M, Σ_1, Σ_2) is an compact 3-dimensional manifold M together with two closed oriented surfaces Σ_1, Σ_2, such that

$$\Sigma_1 \cap \Sigma_2 = \phi, \quad \partial M = (-\Sigma_1) \cup (\Sigma_2). \tag{41}$$

Note that it induces a category whose objects and morphisms are closed surfaces and 3-dimensional cobordisms respectively.

For a given cobordism (M, Σ_1, Σ_2), we assume that the manifold M is triangulated and is given an arrow along each 1-simplex, which induces a triangulation and arrows on Σ_1 and Σ_2. Then we define a $\mathbf{C}$-module V_{Σ_i}, which is freely generated by colorings of Σ_i, for a triangulated surface Σ_i $(i = 1, 2)$ with arrows on all 1-simplices. The dimension of V_{Σ_i} is

$$\dim V_{\Sigma_i} = 2\, l^i\, |X| \tag{42}$$

where l^i is the number of 1-simplices in Σ_i. If $\Sigma = \phi$ then we set $V_{\Sigma_i} = \mathbf{C}$. From now on, we use the symbol Σ_i as a triangulated surface equipped with arrows in the sense we described in previous sections.

We define a homomorphism from V_{Σ_1} to V_{Σ_2} by

$$F_{12}(\alpha_1) \;=\; \sum_{\alpha_2} F(M;\alpha_1\cup\alpha_2)\,\alpha_2 \qquad : V_{\Sigma_1} \to V_{\Sigma_2} \tag{43}$$

where $F(M;\alpha_1\cup\alpha_2)$ is given by eq.(40), α_i is a coloring on Σ_i and the sum is taken over all colorings on Σ_2.

By considering a cobordism $(M;\Sigma_2,\Sigma_1)$ instead of $(M;\Sigma_1,\Sigma_2)$ for the same triangulated manifold M,

$$F_{21}(\alpha_2) \;=\; \sum_{\alpha_1} F(M;\alpha_1\cup\alpha_2)\,\alpha_1 \quad : V_{-\Sigma_2} \to V_{-\Sigma_1}\,. \tag{44}$$

If we identify the vector field V_{Σ_i} with its dual ${V_{\Sigma_i}}^*$, F_{21} is interpreted as a dual map of the linear map F_{12}. Thus, in this sense,

Lemma 1 ${F_{21}}^* \;=\; F_{12}$.

Let $(M;\Sigma_1,\Sigma_3)$ be the composition of cobordisms $(M_1;\Sigma_1,\Sigma_2)$ and $(M_2;\Sigma_2,\Sigma_3)$, i.e., $M=M_1\cup M_2$. We fix a triangulation on M and direction of arrows on 1-simplices. Then

Lemma 2 $F_{23}\,\circ\,F_{12} \;=\; F_{13}$.

Proof.

$$\begin{aligned} F_{23}\,\circ\,F_{12}(\alpha_1) \;&=\; \sum_{\alpha_2} F(M_1;\alpha_1\cup\alpha_2)\,F_{23}(\alpha_2) \\ &=\; \sum_{\alpha_2} F(M_1;\alpha_1\cup\alpha_2)\sum_{\alpha_3} F(M_2;\alpha_2\cup\alpha_3)\,\alpha_3 \end{aligned} \tag{45}$$

where α_i is a coloring on Σ_i. Let N_i^1, N_i^2 and N_i be the numbers of i-simplices of M_1, M_2 and M respectively, and v^j and f^j be the numbers of 0-simplices and 2-simplices of Σ_j. Note that the following relation holds :

$$N_2 = N_2^1 + N_2^2 - f^2, \quad N_0 = N_0^1 + N_0^2 - v^2. \tag{46}$$

Then, by using (40),

$$\begin{aligned} F_{23}\,\circ\,F_{12}(\alpha_1) \;&=\; |X|^{-2N_2^1-N_0^1+f^1+f^2+(v^1+v^2)/2}\,|X|^{-2N_2^2-N_0^2+f^2+f^3+(v^2+v^3)/2} \\ &\quad\times\sum_{\alpha_2}\sum_{\alpha_3}\sum_{\{\phi\}|_{\alpha_1,\alpha_2,\alpha_3}}\prod_{i=1}^{N_3^1}{W_i}^{\kappa_i}\prod_{j=1}^{N_3^2}{W_j}^{\kappa_j} \end{aligned}$$

$$
\begin{aligned}
&= |X|^{-2N_2-N_0+f^1+f^3+(v^1+v^3)/2} \sum_{\{\phi\}|_{\alpha_1,\alpha_3}} \sum_{\alpha_3} \prod_{i=1}^{N_3} W_i{}^{\kappa_i} \\
&= \sum_{\alpha_3} F(M;\alpha_1 \cup \alpha_2 \cup \alpha_3)\,\alpha_3 \\
&= F_{13}(\alpha_1). \qquad \blacksquare
\end{aligned} \tag{47}
$$

Note that a category of **C**-modules whose object and morphism are V_Σ and $F_{ij} : \Sigma_i \to \Sigma_j$ is defined from the above lemma.

For a cobordism $(\Sigma \times I; \Sigma, \Sigma)$ between a triangulated manifold Σ, we denote

$$
F_{id_\Sigma} : V_\Sigma \longrightarrow V_\Sigma \tag{48}
$$

a homomorphism defined by eq.(43).

Lemma 3

$$
tr\ F_{id_\Sigma} = F(\Sigma \times S^1). \tag{49}
$$

Proof.

$$
\begin{aligned}
tr\ F_{id_\Sigma} &= \sum_{\alpha_1=\alpha_2} F(M;\alpha_1 \cup \alpha_2) \\
&= |X|^{-2N_2-N_0+2f+v} \sum_{\{\phi\}|_{\alpha_1=\alpha_2}} \prod_{i=1}^{N_3} W_i{}^{\kappa_i}
\end{aligned} \tag{50}
$$

where v and f is the number of 0-simplices and 2-simplices of Σ. The last equation is equal to $F(\Sigma \times S^1)$ where the manifold $\Sigma \times S^1$ has $N_2 - f$ 2-simplices and $N_0 - v$ 0-simplices. $\blacksquare$

For an arbitrary cobordism $(M; \Sigma_1, \Sigma_2)$, it is shown that

$$
F_{12} = F_{12} \circ F_{id_{\Sigma_1}} \tag{51}
$$

is satisfied from proposition 2 and lemma 2. Thus

$$
Ker\,(F_{12}) \supset Ker\,(F_{id_{\Sigma_1}}). \tag{52}
$$

Furthermore, the equation

$$
Im(F_{12}) = Im(F_{12})/Ker(F_{id_{\Sigma_2}}) \tag{53}
$$

follows from

$$
F_{12} = F_{id_{\Sigma_2}} \circ F_{12}. \tag{54}
$$

By the equations (52) and (53), the map

$$\Psi_{12} \,:\, Q_{\Sigma_1} \longrightarrow Q_{\Sigma_2} \tag{55}$$

is induced by a homomorphism $F_{12} : V_{\Sigma_1} \to V_{\Sigma_2}$ where

$$Q_{\Sigma_i} \;=\; V_{\Sigma_i} / \, Ker(F_{id_{\Sigma_i}}). \tag{56}$$

Note that the map $\Psi_{id_\Sigma} : Q_\Sigma \to Q_\Sigma$ corresponding to F_{id_Σ} is a monomorphism and can be regarded as identity map on Q_Σ if we choose the basis of Q_Σ properly. Thus the correspondence $\Sigma \to Q_\Sigma$ and $(M; \Sigma_1, \Sigma_2) \to \Psi_{12}$ forms a functor from the category of cobordisms with triangulation and arrows to the category of $\mathbf{C}$-modules.

Furthermore, by considering a map

$$\Psi_{id_\Sigma} = \Psi_{\Sigma\Sigma'} \circ \Psi_{\Sigma'\Sigma}$$

for a manifold $M = \Sigma \times I$ with $\partial M = (-\Sigma) \cup \Sigma'$ where the topology of the two triangulated surfaces Σ and Σ' are the same, we can show the relation

$$\dim\, Q_\Sigma = \dim\, Q_{\Sigma'}. \tag{57}$$

We identify Q_Σ and $Q_{\Sigma'}$ by means of the map $\Psi_{\Sigma\Sigma'} : Q_\Sigma \to Q_{\Sigma'}$. Thus from this identification, the map $\Psi_{\Sigma\Sigma'} : Q_\Sigma \to Q_{\Sigma'}$ is independent of the choice of triangulation and arrows on Q_Σ and $Q_{\Sigma'}$.

The above arguments defines a functor from the 3-dimensional cobordisms of non-triangulated surfaces to the category of $\mathbf{C}$-modules: $\Sigma \to Q_\Sigma$ and $(M; \Sigma_1, \Sigma_2) \to \Psi_{12}$. Thus combining these results with lemma 1 and lemma 2,

Theorem 2 *The functor defined as above is nothing but a functor of three-dimensional topological quantum field theory which satisfies the Atiyah's axioms [1].*

The important consequence of the axioms of topological quantum field theory is

$$\dim Q_\Sigma \;=\; F(\Sigma \times S^1), \tag{58}$$

which is straightforward from lemma 3.

Remember that (16) and (17), i.e., for any choice of involutory Hopf algebra A,

$$Z_A(S^2 \times S^1) = 1, \qquad Z_A(S^3) = |X|^{-1}\,.$$

Then, we see from eq.(58)

$$\dim Q_{S^2} = 1, \tag{59}$$

which, with lemma 2, leads to

$$Z_A(M)Z_A(S^3) = Z_A(M_1)Z_A(M_2) \tag{60}$$

where $M = M_1 \# M_2$.

7 Generalization

In this section, we generalize the topological invariant $Z_A(M)$ $(= F(M))$ to that of compact manifold M with boundary, i.e., we give a topological invariant complex number to M.

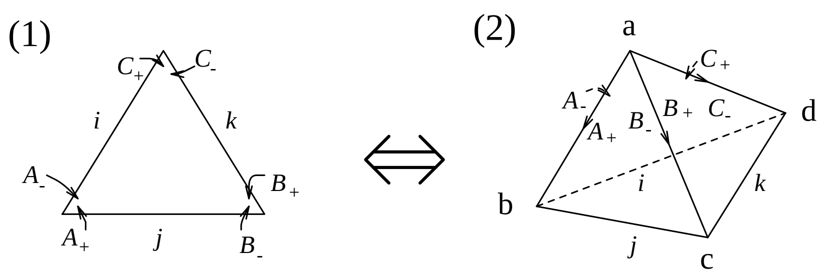

Figure 8: A triangle with color. The weight $\tilde{W}_F{}^\kappa$ defined for a colored triangle F is given by the weight W^κ of the tetrahedron T depicted in the right-hand side. For any F, we set $a > b$, $a > c$, $a > d$.

Definition 1 *A vertex coloring on the boundary ∂M of a triangulated manifold M is a map*

$$\chi : \{(V_{i_0}, E_{i_1}, F_{i_2}) \,|\, i_k = 1, \cdots, M_k (k = 0, 1, 2),\ E_{i_1} \cap F_{i_2} = E_{i_1},\ V_{i_0} \cap E_{i_1} = V_{i_0}\} \longrightarrow X \tag{61}$$

where V_{i_0}, E_{i_1} and F_{i_2} denote 0-, 1- and 2-simplex in ∂M respectively and M_k is the number of k-simplices in ∂M (Fig.8(1)).

To give a vertex coloring is equivalent to give $6 \times M_2$ $(= 4 \times M_1)$ symbols, which are elements of X, to all triples of eq.(61). We assume that all 1-simplices in M are equipped with arrows induced by an order of vertices as in section 4.

Now we define a weight $\tilde{W}_F$ for a triangle $F \in \partial M$ with ordered vertices b, c and d. We temporarily put a new vertex a in the outside of F as $a \notin M$ as $a > b$, $a > c$ and $a > d$ Then we make a tetrahedron T of a, b, c, d whose coloring is induced by that of F as Fig.8. We define the weight $\tilde{W}_F{}^{\tilde{\kappa}}$ of F as the weight W^κ of the tetrahedron T given above :

$$\tilde{W}_F{}^{\tilde{\kappa}} \begin{pmatrix} i_+, i_- & j_+, j_- & k_+, k_- \\ A_+, A_- & B_+, B_- & C_+, C_- \end{pmatrix} \equiv W^\kappa \begin{pmatrix} i_+, i_- & j_+, j_- & k_+, k_- \\ A_+, A_- & B_+, B_- & C_+, C_- \end{pmatrix} \tag{62}$$

where $i_\pm$, $j_\pm$ and $k_\pm$ are colors on edges of a triangle F and $A_\pm$, $B_\pm$ and $C_\pm$ are colors on vertices on F (Fig.8). From the assignment of the weight $\tilde{W}_{F_i}{}^{\tilde{\kappa}_i}$ for each triangle F_i which belongs to the boundary ∂M of M in addition to that of $W_i{}^{\kappa_i}$ for each 3-simplices, the following quantity is defined :

$$\tilde{F}(M) = |X|^{-2N_2 - N_0 - 2M_1} \sum_\phi \prod_{i=1}^{N_3} W_i{}^{\kappa_i} \sum_\chi \prod_{j=1}^{M_2} \tilde{W}_{F_j}{}^{\tilde{\kappa}_j} \in \mathbf{C}. \tag{63}$$

Theorem 3 *$\tilde{F}(M)$ is independent of the direction of arrows on 1-simplices of M and any local topology preserving deformation of M.*

Proof.

Note that $\tilde{F}(M)$ is equal to $Z_A(M)$ in the case of M being a closed manifold. Therefore the invariance of $\tilde{F}(M)$ under the local deformation of M which does not change the triangulation of ∂M is verified by applying Prop.2. Furthermore, the independence of direction of arrows on 1-simplices in M is easily shown by the relation

$$\Delta^{x_1 x_2 \cdots x_n} = \Delta^{y_n y_{n-1} \cdots y_1} S^{x_1}{}_{y_1} S^{x_2}{}_{y_2} \cdots S^{x_n}{}_{y_n}. \tag{64}$$

Thus we have only to show the invariance of $\tilde{F}(M)$ under the local deformation which changes the triangulation on the boundary. Remember that all topology preserving moves of a triangulated surface is generated by finite sequence of $(2,2)$-moves, $(3,1)$-moves and $(1,3)$-moves [14]. In the case of a triangulated surface that is a boundary of a triangulated 3-manifold M, the moves are induced by adding a new 3-simplex to the boundary. We provide two types of moves. One is a '$(2,2)$-move' on ∂M induced by the addition of two new triangles, and thus a new tetrahedron, to M as Fig.9(1). The other is a '$(3,1)$-move'

induced by adding a triangle to form a new tetrahedron on M as Fig.9(2). Note that under these moves the numbers N_2, N_0 and M_1 are changed as

$$\text{`}(2,2)-\text{move}' \qquad (N_2, N_0, M_1) \longrightarrow (N_2+2, N_0, M_1), \tag{65}$$

$$\text{`}(3,1)-\text{move}' \qquad (N_2, N_0, M_1) \longrightarrow (N_2+1, N_0, M_1-3). \tag{66}$$

These two types of moves is sufficient for all topology preserving deformation on ∂M because $(1,3)$-moves are generated by the inverse operation of the '$(3,1)$-move' described above, i.e., removing a triangle F from ∂M after reforming a triangulation of $M/\partial M$ so that every edge of F is a boundary of at least three triangles. The invariance of $\tilde{F}(M)$ under '$(2,2)$-moves' and '$(3,1)$-moves' is verified explicitly by using the relations (37) and (38) which describe the '$(2,2)$-move' and the '$(3,1)$-move' respectively as follows :

'$(2,2)$-move' :

$$\begin{aligned}
&|X|^4 \tilde{W}^+ \begin{pmatrix} i_1, i_2 & j_1, j_2 & k_1, k_3 \\ B_3, B_2 & D_1, D_3 & C_1, C_2 \end{pmatrix} \tilde{W}^- \begin{pmatrix} l_1, l_2 & m_1, m_2 & k_3, k_2 \\ D_3, D_2 & B_1, B_3 & A_1, A_2 \end{pmatrix} \\
&= \tilde{W}^+ \begin{pmatrix} n_1, n_2 & j_3, j_2 & l_3, l_2 \\ B_1, B_2 & A_1, A_3 & C_1, C_3 \end{pmatrix} \\
&\qquad \times W^+ \begin{pmatrix} i_3, i_2 & j_1, j_3 & k_1, k_2 \\ l_1, l_3 & m_3, m_2 & n_2, n_3 \end{pmatrix} \tilde{W}^- \begin{pmatrix} n_3, n_1 & m_1, m_3 & i_1, i_3 \\ D_1, D_2 & C_3, C_2 & A_3, A_2 \end{pmatrix}
\end{aligned} \tag{67}$$

'$(3,1)$-move' :

$$\begin{aligned}
&|X|^{-4} \tilde{W}^+ \begin{pmatrix} t_3, t_2 & u_1, u_3 & j_1, j_2 \\ C_1, C_3 & A_3, A_2 & D_1, D_2 \end{pmatrix} \\
&\qquad \times \tilde{W}^+ \begin{pmatrix} s_3, s_2 & t_1, t_3 & k_1, k_2 \\ A_1, A_3 & B_3, B_2 & D_2, D_3 \end{pmatrix} \tilde{W}^- \begin{pmatrix} s_1, s_3 & i_1, i_2 & u_3, u_2 \\ C_3, C_2 & D_3, D_1 & B_1, B_3 \end{pmatrix} \\
&= \tilde{W}^+ \begin{pmatrix} i_3, i_2 & j_3, j_2 & k_3, k_2 \\ A_1, A_2 & B_1, B_2 & C_1, C_2 \end{pmatrix} W^+ \begin{pmatrix} t_1, t_2 & u_1, u_2 & j_1, j_3 \\ i_1, i_3 & k_1, k_3 & s_1, s_2 \end{pmatrix} \quad .
\end{aligned} \tag{68}$$

These relations with (65) and (66) ensure that the value $\tilde{F}(M)$ of eq.(63) does not change under '$(2,2)$-move' and '$(3,1)$-move.' Since all topology preserving deformations of M is

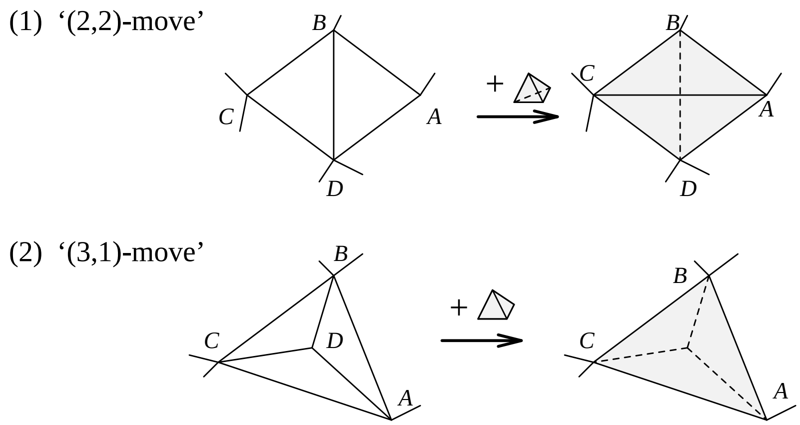

Figure 9: '$(2,2)$-move' and '$(3,1)$-move' on the boundary ∂M of M. The addition of a tetrahedron to ∂M generates the local move, (1) or (2), on ∂M.

generated by these moves and their inverses together with the local deformations of M which do not change the triangulation of ∂M, the proof is completed. ■

The following formula is verified from the definition :

$$\tilde{F}(M \setminus \mathrm{Int}\, D^3) = |X| \tilde{F}(M) \tag{69}$$

where D^3 denotes the three-dimensional ball.

8 Concluding Remarks

In this paper, we generalized the Chung-Fukuma-Shapere invariant $Z_A(M)$ of closed three-dimensional manifold M to the functor Ψ_{ij} of a topological quantum field theory which satisfies Atiyah's axioms. We also generalized $Z_A(M)$ to the invariant $\tilde{F}(M)$ of a compact manifold M with boundary.

The crucial point of defining the functor and $\tilde{F}(M)$ for a triangulated manifold M is to give the weight ${W_i}^{\kappa_i}$ to each tetrahedron T_i. The weight in this theory plays the same role as that of quantum $6j$-symbols in the Turaev-Viro theory. After defining ${W_i}^{\kappa_i}$, it is straightforward to define the functor and $\tilde{F}(M)$ by referring to the Turaev-Viro theory and its generalization in ref. [12]. Note that the weight ${W_i}^{\kappa_i}$ is defined for colors on every pair of adjacent *edge* and *face* of T_i with arrows. On the other hand, quantum $6j$-symbol is defined for colors on edges without arrows.

Finally, we give two remarks; we can define the functor and $\tilde{F}(M)$ for any cell complex which $Z_A(M)$ is defined on, though we do not give the definition; Kuperberg generalized his invariant, which is equivalent to $Z_A(M)$, for non-involutory Hopf algebras [11].

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